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Modeling Inflation and Money Demand Using a Fourier-Series Approximation

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1. Introduction

Consider the economic time-series model given by:

$$y_t = \alpha_t + \beta x_t + \varepsilon_t \quad (1)$$

where: α_t is the time-varying intercept, x_t is a vector containing exogenous explanatory variables and/or lagged values of y_t, and ε_t is an *i.i.d.* disturbance term that is uncorrelated with any of the series contained in x_t. The notation in (1) is designed to emphasize the fact that the intercept term is a function of time. Although it is possible to allow the value of β to be time-varying, in order to highlight the effects of structural change, we focus only on the case in which the intercept changes. If the functional form of α_t is known, the series can be estimated, hypotheses can be tested and conditional forecasts of the various values of $\{y_{t+j}\}$ can be made. In practice, two key problems exist; the econometrician may not be sure if there is parameter instability and, if such instability exists, what form it is likely to take. Parameter instability could result from any number of factors including structural breaks, seasonality of an unknown form and/or an omitted variable from a regression equation.

The time-series literature does address the first problem in great detail. In addition to the standard Chow (1960) test and Hausman (1978) test, survey articles by Rosenberg (1973) and Chow (1984) discuss numerous tests designed to detect structural change. More recently, Andrews (1993) and Andrews and Ploberger (1994) have shown how to determine if there is a one-time change in a parameter when the change point is unknown, Hansen (1992) has considered parameter instability in regressions containing $I(1)$ variables, Lin and Teräsvirta

(1994) showed how to test for multiple breaks, and Tan and Ashley (1999) formulated a test for frequency dependence in regression parameters.

The second problem is more difficult to address since there are many potential ways to model a changing intercept when the functional form of α_t is unknown. For example, it is possible to include dummy variables to capture seasonal effects or to represent one or more structural breaks. Similarly, the inclusion of additional explanatory variables may capture the underlying reason for the change in the intercept. The time-varying intercept may be estimated using a Markov-switching process or a threshold process. Yet another avenue for exploration is to let the data determine the functional form of α_t. For example, the local-level model described in Harvey (1989) uses the Kalman Filter to estimate α_t as an autoregressive (or unit-root) process. The purpose of this paper is to demonstrate how the misspecification problem can be alleviated by the use of a methodology that 'backs-out" the form of time-variation. The modeling strategy is based on a Fourier approximation in that it uses trigonometric functions to approximate the unknown functional form.

The choice of the Fourier approximation as the method for modeling the time-varying intercept is driven by two major considerations. *First*, it is well-known that a Fourier approximation can capture the variation in any absolutely integrable function of time. Moreover, there is increasing awareness that structural change may often be gradual and smooth (Leybourne et al., 1998, Lin and Teräsvirta, 1994), rather than the sudden and discrete changes that are usually modeled by conventional dummy variables. As will become apparent, the Fourier approximation is particularly adept at modeling this kind of time variation. *Second*, the Fourier approach needs no prior information concerning the actual form of the time-varying intercept α_t. Traditional models using dummy variables or more recent developments based on nonlinear deterministic time trends (Ripatti and Saikkonen, 2001) require that the form of the time variation be specified at the outset. There is also a need to discriminate among alternative specifications using standard diagnostic tools. As noted by Clements and Hendry (1998, pp. 168-9), parameter change appears in many guises and can cause significant forecast error in practice. They also establish that it can be difficult to distinguish model misspecification from the problem of non-constant parameters.

The use of the Fourier approximation is now well established in the econometric literature as Gallant (1984), Gallant and Souza (1991), and Becker, Enders and Hurn (2004) use one or two frequency components of a Fourier approximation to mimic the behavior of an unknown functional form. Moreover, the problem of testing for trigonometric components with predetermined frequencies was tackled by Farley and Hinich (1970, 1975) in the context of a model with parameter trend. Similarly, a test for the significance of trigonometric terms in a regression equation with an *unknown* frequency component was introduced by Davies (1987). In fact, Davies' (1987) results are an important building block in our methodology. Davies' test is analogous to that of Tan and Ashley (1999) if their frequency band is restricted to a single frequency.

There are many tests for parameter instability and it is not the intention of this paper to merely present the empirical properties of yet another. Instead, our proposed methodology is intended to be most helpful when it is not clear how to model the time-varying intercept. The novel feature of this approach is that it uses the time-varying intercept as a modeling device to capture the form of any potential structural breaks and, hence, lessen the influence of model misspecification.

The rest of the paper is structured as follows. Section 2 makes the simple point that a low-order Fourier approximation can mimic a time-varying intercept term. Davies' (1987) method of selecting a single frequency component and testing its statistical significance is presented in detail. Section 3 illustrates the methodology using the U.S. inflation rate. In particular, we show that a linear specification is inappropriate since the intercept for the 1970's and 1980's is high relative to the rest of the sample period. Section 4 describes a method to select multiple frequency components so as to mimic the form of the time-varying intercept. In Section 5, we estimate the demand for money (as measured by M3). In essence, we back-out the form of the so-called "missing money." It is particularly interesting that the time-varying intercept suggests that money demand was never a stable function of the price level, real income and the short-term interest rate. There is the strong suggestion that the missing money has the same form as the major stock market indices. Conclusions and limitations of our work are discussed in the final section.

2. Modeling with a Fourier Approximation

If α_t is an absolutely integrable function of time, for any desired level of accuracy, it is possible to write:[1]

$$\alpha_t = A_0 + \sum_{k=1}^{s}\left[A_k \sin\frac{2\pi k}{T}\bullet t + B_k \cos\frac{2\pi k}{T}\bullet t \right] \quad ; \quad s \leq T/2 \tag{2}$$

where: s refers to the number of frequencies contained in the process generating α_t, k represents a particular frequency and T is the number of usable observations.

Figure 1 illustrates the simple fact that use of a single frequency in a Fourier approximation can approximate a wide variety of functional forms. The solid line in each of the four panels represents a sequence that we approximate using a single frequency. We let the four panels depict sharp breaks since the smooth Fourier approximation has the most difficulty in mimicking such breaks. Consider Panel *a* in which the solid line represents a one-time change in the level of a series containing 100 observations ($T = 100$). Notice that a single frequency such that $\alpha_t = 2.4 - 0.705 sin(0.01226\ t) - 1.82 cos(0.01226\ t)$ captures the fact that the sequence increases over time (Note: $k = 0.1953$ and $2\pi*0.1953/100 = 0.01226$). In Panel *b,* there are two breaks in the series. In this case, the approximation $\alpha_t = 0.642 - 0.105 sin(0.586\ t) - 0.375 cos(0.586\ t)$ captures the overall tendency of the series to increase. The solid line in Panel *c* depicts a sequence with a temporary change in the level while the solid line in Panel *d* depicts a "seasonal" sequence that is low in periods 1 – 25 and 51 – 75 and high in periods 26 – 50 and 76 – 100. Again, the approximations using a single frequency do reasonably well. It is interesting that the frequency used for the approximation in Panel *d* is exactly equal to 2.0 since there are two regular changes in the level of the sequence.

Note that the approximation can be improved by using more than one frequency. Suppose that the solid line in Figure 2 represents a sequence that we want to estimate. If we approximate this sequence with a single frequency ($k = 1.171$), we obtain the dashed line labeled "1 Frequency." If we add another frequency component using $k_1 = 1.171$ and $k_2 = 2.72$, the approximation is now depicted by the line labeled "2 Frequencies" in Figure 2.

Thus, each of these sequences can be approximated by a small number of frequency components. The point is that the behavior of any deterministic sequence can be readily captured by a sinusoidal function even though the sequence in question is not periodic. As such, the intercept may be represented by a deterministic time-dependent coefficient model without first specifying the nature of the nonlinearity. Since it is not possible to include all frequencies in (2), the specification problem is to determine which frequencies to include in the approximation. As a practical matter, the fact that we use a small number of frequencies means that the Fourier series cannot capture all types of breaks. Figures 1 and 2 suggest that our Fourier approximation will work best when structural change manifests itself smoothly.

Davies (1987) shows how to select the most appropriate single frequency and to test its statistical significance. Suppose the $\{\xi_t\}$ sequence denotes an *i.i.d.* error process with a unit variance. Consider the following regression equation:

$$\xi_t = A_k \sin(2\pi kt/T) + B_k \cos(2\pi kt/T) + e_t \tag{3}$$

where: A_k and B_k are the regression coefficients associated with the frequency k.

For any value of k, it should be clear that rejecting the null hypothesis $A_k = B_k = 0$ is equivalent to rejecting the hypothesis that the $\{\xi_t\}$ sequence is *i.i.d.* Since the frequency k is unknown, a test of the null hypothesis involves an unidentified nuisance parameter. As such, it is not possible to rely on standard distribution theory to obtain an appropriate test statistic. Instead, if $S(k)$ is the test statistic in question, Davies uses the supremum:

$$S(k^*) = sup\{S(k): L \leq k \leq U\} \tag{4}$$

where: k^* = is the value of k yielding the largest value of $S(k)$ and $[\,L,\ U\,]$ is the range of possible values of k.

Davies reparameterizes (3) such that:

$$E_{t-1}(\xi_t) = a_1\sin[\,(t - 0.5T - 0.5)\theta\,] + b_1\cos[\,(t - 0.5T - 0.5)\theta\,] \tag{5}$$

where: $\theta = 2\pi k/T$ so that the values of $\{\xi_t\}$ are zero-mean, unit-variance *i.i.d.* normally distributed random variables with a period of oscillation equal to $2\pi/k$ (since $\theta = 2\pi k/T$).

For the possible values of θ in the range $[\,L, U\,]$ where $0 \leq L < U \leq \pi$, construct:

$$S(k) = \left[\sum_{t=1}^{T} \xi_t \sin[(t - 0.5T - 0.5)\theta]\right]^2 / v_1 + \left[\sum_{t=1}^{T} \xi_t \cos[(t - 0.5T - 0.5)\theta]\right]^2 / v_2 \tag{6}$$

where: $v_1 = 0.5T - 0.5\sin(T\theta)/\sin(\theta)$ and $v_2 = 0.5T + 0.5\sin(T\theta)/\sin(\theta)$.

Davies shows that:

$$prob\ [\ \{ S(k^*) : L \leq \theta \leq U \} > u\ ] \tag{7}$$

can be approximated by:[2]

$$Tu^{0.5}e^{-0.5u}(U - L)/(24\pi)^{0.5} + e^{-0.5u} \tag{8}$$

Given T, U and L, critical values for $S(k^*)$ can be derived from equations (7) and (8). Note that Davies' method is equivalent to estimating (3) for each possible frequency in the interval $0 < U - L \leq T/2$. The frequency providing the smallest residual sum of squares is the same k^* yielding the supremum $S(k^*)$. It is this value of k^* that is a candidate for inclusion in the time-varying intercept.

Becker, Enders and Hurn (2004) discuss a modified test version of the Davies (1987) test that can be used in a regression framework. Let the data generating process be given by $y_t = \beta_0 + \varepsilon_t$. To test for a structural break in the intercept, estimate the following regression equation by ordinary least squares (OLS) for each potential frequency k:

$$y_t = \beta_0 + \beta_1\sin(2k\pi t/T) + \beta_2\cos(2k\pi t/T) + \varepsilon_t \tag{9}$$

Let the value k^* correspond to the frequency with the smallest residual sum of squares, RSS^*, and let β_1^* and β_2^* be the coefficients associated with k^*. Since the trigonometric components are not in the data-generating process, β_1^* and β_2^* should both equal zero. However, the usual F-statistic for the null hypothesis $\beta_1^* = \beta_2^* = 0$ does not follow a standard distribution since the coefficients are estimated using a search procedure and k^* is unidentified under the null hypothesis of linearity. The critical values depend on the sample size and the maximum frequency used in the search procedure; the critical values for the OLS procedure are reproduced in Table 1. Note that this is a supremum test since k^* yields the minimum residual sum of squares.

2.2 Dependent error structures

It is not straightforward to modify the Davies test or the *Trig*-test for the case of a dependent error process. Nevertheless, Enders and Lee (2004) develop a variant of the *Trig*-test when the errors have a unit root. Suppose that $\{y_t\}$ is the unit-root process: $y_t = \beta_0 + \mu_t$, where $\mu_t = \mu_{t-1} + \varepsilon_t$ and that the researcher estimates a regression equation in the form of (9) by ordinary least squares (OLS) for each potential frequency k. Enders and Lee (2004) derive the asymptotic distribution of the F-test for the null hypothesis $\beta_1^* = \beta_2^* = 0$. They tabulate critical values for sample sizes of 100 and 500 searching over the potential frequencies to obtain the one with the best fit (k^*). As in Becker, Enders and Hurn (2004), their tabulated critical values, called $F(k^*)$, depend on sample size and the maximum frequency used in the search procedure. It should be clear that the $F(k^*)$ test is a supremum test since k^* yields the minimum residual sum of squares. For a sample size of 100 using a maximum value of $k = 10$, Enders and Lee (2004) report the critical values of $F(k^*)$ to be 10.63, 7.78 and 6.59 at the 1%, 5%, and 10% significance levels, respectively.

2.2 Power

Four conclusions emerged from Davies' small Monte Carlo experiment concerning the power of his test. *First*, for a number of sequences with structural breaks, the power of the test increases in the sample size T. *Second*, the power of the test seems to be moderately robust to non-normality. *Third*, if the frequency is not an integer, the use of integer frequencies entails a loss of power. *Fourth*, if the frequency is an integer, the power of the discrete form of the test exceeds that of the test using fractional frequencies. Moreover, as can inferred from equations

(7) and (8), increasing the size of U - L increases the probability of any given value of u. Thus, unnecessarily expanding the size of the interval will reduce the power of the test. Since we are considering a small number of structural breaks, it makes sense to use a small value of U since a structural break is a 'low frequency' event.

It is well known that the most powerful test for a one-time change in the mean is that of Andrews and Ploberger (AP) (1994). To further illustrate the power of Davies' test, we performed our own Monte Carlo analysis using equation (1) such that:

x_t and $\varepsilon_t \sim N(0,1)$, $\beta = 1$ and:

$$\alpha_t = \begin{cases} 0, & \forall t \leq 40 \\ \delta, & \forall t > 40 \end{cases} \tag{10}$$

We considered values of k in the range [0, 1] in order to allow for the possibility of an infrequent change in the mean. After all, a frequency greater than one is not likely to replicate a single break. Table 2 shows the power of the AP and the Davies tests for different break sizes δ.

Of course, if it is known that there cannot be more than a single break in the intercept, the AP test is preferable to the Davies test. However, the Davies test does perform almost as well as the optimal test for a single break. We performed a second Monte Carlo experiment to validate the notion that a Fourier approximation can be especially useful to mimic a sequence with multiple breaks. As such, we modified the data generating process in (9) to have a second structural break:

$$\alpha_t = \begin{cases} 0, & t \leq 20 \\ \delta, & 20 < t \leq 40 \\ 0, & t > 40 \end{cases} \tag{11}$$

As shown in Table 3, the Davies test still possesses reasonably high power, while the AP test has much weaker power compared to its power against a one time structural break.[3] For reasonably sized values of δ, the power of the Davies test exceeds that of the AP test.

Finally, Becker, Enders and Hurn (2004) show that Davies' test and their modification of the Davies' test (called the *Trig*-test) can have more power than the Bai-Perron (1998) test when the number of breaks is unknown. They show that the Davies test and the *Trig*-test have the correct empirical size and excellent power to detect structural breaks and stochastic parameter variation of unknown form.

3. A Structural Break in the Inflation Rate

To illustrate the use of the test for a single frequency component, we update and extend the example of Becker, Enders and Hurn (2004). We consider the application of the test to multiple frequencies in Section 4. In order to use the test it is necessary to standardize the residuals to have a unit variance.[4] A more important issue is that regression residuals are only estimates of the actual error process. Hence, an alternative to obtaining critical values from (7) and (8) is to bootstrap the $S(k^*)$ statistic. In order to illustrate the use of Davies' test, we obtained monthly values of the U.S. *CPI* (seasonally adjusted) from the website of the Federal Reserve Bank of St. Louis (http://www.stls.frb.org/fred/index.html) for the 1947:1 to 2004:8 period. It is well known that inflation rates, measured by the *CPI*, act as long-memory processes. For example, Baillie, Han and Kwon (2002) review a number of papers indicating that U.S. inflation is fractionally integrated and Clements and Mizon (1991) argue that structural breaks can explain such findings; a break in a time-series can cause it to behave like a unit-root process.

If we let π_t denote the logarithmic change in the U.S. *CPI*, the following augmented Dickey-Fuller test (with *t*-statistics in parentheses) shows that the unit-root hypothesis can be rejected for our long sample:[5]

$$\Delta\pi_t = \underset{(3.17)}{0.603} - \underset{(-4.35)}{0.173}\pi_{t-1} + \sum_{i=1}^{11}\beta_i\,\Delta\pi_{t-i} + e_t \qquad (12)$$

The key point to note is that standard diagnostic checks of the residual series $\{e_t\}$ indicate that the model is adequate. If ρ_i denotes the residual autocorrelation for lag *i,* the correlogram is:

ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}	ρ_{11}	ρ_{12}
-0.007	0.013	0.034	-0.008	0.002	0.051	-0.019	0.012	0.028	-0.012	-0.039	0.077

However, when we performed a Dickey-Fuller test using a more recent sample period (1973:1 - 2004:8), the unit-root hypothesis cannot be rejected. Consider:

$$\Delta\pi_t = \underset{(1.72)}{0.440} - \underset{(-2.08)}{0.095}\pi_{t-1} + \sum_{i=1}^{11}\beta_i\,\Delta\pi_{t-i} + e_t \qquad (13)$$

In order to determine why the unit-root hypothesis is rejected over the entire sample period but not the latter period, we performed additional diagnostic checks on (12). For example, the RESET test suggests that the relationship is nonlinear. Let $\Delta\hat{\pi}_t$ and $\hat{\varepsilon}_t$ denote the fitted values and the residual values of equation (12), respectively. We regressed $\hat{\varepsilon}_t$ on all of the 'explanatory' variables in (12) *and* on $\Delta\hat{\pi}_t^H$. The idea of the RESET test is that this regression should have little explanatory power if the actual data generating process is linear. For values of *H* equal to 3, 4 and 5, the *prob*-values for the RESET test are 0.011, 0.002 and 0.000, respectively. Moreover, Hansen's (1992) test for parameter instability has a *prob*-value that is less than 0.01. Thus, both tests suggest that some form of nonlinearity might be present in (12). However, neither test suggests the nature of the nonlinearity.

We standardized the residuals from (12) and, since we are searching for a small number of breaks, constructed the values of $S(k)$ for integer frequencies $k = [\ 1, 8\ ]$.[6] The "best" fitting frequency was found to 1.00 and the sample value $S(k^*) = 11.02$. If we use Davies' critical values, this value of $S(k^*)$ has a *prob*-value of less than 1%. Our concern about the use of estimated error terms led us to bootstrap the $S(k^*)$ statistic using the residuals from (12). We found that 95% of the bootstrapped values of $S(k^*)$ exceeded 5.94 and 99% exceeded 8.82. Hence, there is clear evidence of a structural break in the inflation rate. Next, using $k^* = 1.0$, we estimated the regression equation:[7]

$$\Delta\pi_t = \underset{(4.89)}{1.08} - \underset{(-1.87)}{0.330}\sin(2\pi t/T) - \underset{(-4.06)}{0.803}\cos(2\pi t/T) - \underset{(-6.01)}{0.301}\pi_{t-1} + \sum_{i=1}^{11}\beta_i\,\Delta\pi_{t-i} + \varepsilon_t \qquad (14)$$

The time path π_t is shown in Panel *a* of Figure 3 and the time path of 1.08 - 0.330 $\sin(2\pi t/T)$ - 0.803 $\cos(2\pi t/T)$ is shown in Panel *b*. It is clear from examining the time-varying intercept, that the period surrounding the 1970's and 1980's is different from the other periods. Such a structural break can explain why the results of the Dickey-Fuller tests differ over the two sample periods. If we wanted to refine the approximation of the time-varying intercept, we could apply the test a second time. However, our aim has been to illustrate the use of the Davies' test for modeling a break using a single frequency. The appropriate selection of multiple frequency components is addressed in the next section.[8]

4. Selecting the optimal number of terms in the Fourier expansion

The Davies' test and the *Trig*-test are appropriate when the null hypothesis is that the regression residuals are *i.i.d*. At the other extreme, the test of Enders and Lee (2004) is for the case of a nonstationary error process. Note that all three papers test for the presence of a single frequency component. Our aim is a bit different in that we seek to select multiple frequencies in situations where the null hypothesis may not be that of a unit root or *i.i.d.* errors. Hence, one difficulty we face is that the selection of multiple frequencies can entail problems concerning sequential testing. As discussed in Hendry (1995) and Davidson (2000), sequential testing may cause differences between the actual and the nominal size of the test, even if the individual tests have the correct size. The second problem we face involves the issue of dependent errors since there is no test for the presence of frequency components under the general case of stationary, but not necessarily *i.i.d.*, errors. It might seem reasonable to use the block bootstrap of Künsch (1989) or the stationary bootstrap of Politis and Romano (1994) to sequentially test each frequency component to be included in the intercept. After all, Li and Maddala (1996) and Hansen (1999) indicate that bootstrapping methods can be applied in the presence of unidentified nuisance parameters. The problem is that these bootstrapping procedures are designed to replicate the autocorrelation pattern in the residuals as a feature of the model under the null hypothesis. Structural breaks in the intercept term, however, will tend to manifest themselves in the residual autocorrelations of the restricted model. As such, the power to detect significant trigonometric terms would necessarily be extremely small.

Our proposed method attempts to circumvent these two problems when selecting multiple frequencies. When the null hypothesis is that the errors are *i.i.d.* (as in the previous example concerning the inflation rate), it is possible to bootstrap individual and/or groups of selected frequency components. Thus, the reliance of multiple applications of the $S(k^*)$ statistic is avoided. When the null hypothesis does not require unit root or *i.i.d.* errors, bootstrapping the individual frequency components becomes problematic. Instead, we sequentially add frequencies to (2) as long as one of the model selection criteria, such as the *AIC* or *BIC,* continues to decline. Our own preference is to use the *BIC* since it will select the more parsimonious model. At each step, the frequency that maximizes the statistic, $S(k^*)$ in (6) is chosen.[9] Once all such frequencies are chosen (so that the *BIC* is as small as possible), we test the null hypothesis that all values of $A_k = B_k = 0$ by bootstrapping. We conjecture that bootstrapping is feasible since Enders and Lee (2004) show that the $F(k^*)$ statistic can be derived and tabulated even in the case of nonstationary errors.[10] In summary, we select frequencies sequentially using Davis (1987) grid search method and the number of frequency components is selected by the *BIC*. We then bootstrap the joint test that all frequency components are equal to zero. Unfortunately, the nature of the bootstrapping method that is appropriate for one application may not be appropriate for the next. As such, we illustrate the method for the difficult case wherein estimated equation is thought to be a cointegrating relationship.

5. Structural Breaks in the Demand for Money

As discussed in a number of survey articles, including those by Goldfeld (1976) and Judd and Scadding (1982), there is a vast literature indicating a breakdown in the simple money demand relationship. As such, it seemed reasonable to apply our methodology to see if it could facilitate the modeling of a notorious problem. Consequently we obtained quarterly values of the U.S. money supply as measured by M3, seasonally adjusted real and nominal GDP, and the 3-month treasury bill rate for the period 1959:1 – 2004:2 from the website of the Federal Reserve Bank of St. Louis (www.stls.frb.org/index.html).[11] We constructed the price level as the ratio of nominal to real GDP. As shown in Table 4, augmented Dickey-Fuller tests including a time trend in the estimating equation indicated that the logarithms of M3 (m), real GDP (y), and the price level (p) do not act as trend stationary processes. Even though the trend was excluded for the interest rate, the 3-month *T*-bill rate (r) does not seem to exhibit any mean reversion.

We then estimated the simple money demand function (with t-statistics in parentheses):

$$m_t = -0.128 + 1.01p_t + 1.10y_t + 0.005r_t \qquad (15)$$
$$(-2.52)\ (24.73)\quad (19.77)\quad (2.54)$$

$AIC = -41.50$, $BIC = -28.69$

Although the price and income elasticities are statistically significant and are of the correct sign and magnitude, there are some serious problems with the regression equation. In addition to the fact that the interest rate semi-elasticity of demand is positive, the residuals are not well-behaved. For example, the autocorrelations of the residuals are quite high:

ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8
0.98	0.94	0.90	0.85	0.80	0.75	0.69	0.63

The impression that (15) is not a cointegrating vector is confirmed by the Engle-Granger (1987) test. Both the *AIC* and *BIC* selected a lag length of one. For this lag length, the t-statistic for the null hypothesis that the variables are not cointegrated is only –1.86.

Of course, a structural break or a missing variable may be one reason that the residuals of (15) appear to be nonstationary. At this point, it is not our aim to determine whether the residuals pass a test for white-noise. Equation (15) requires only that the residuals be $I(0)$ so that it is not appropriate to use the Davies test. Instead, we want to determine the most appropriate frequency to include in our Fourier approximation of the intercept term. We used the standardized residuals $\{\xi_t\}$ to construct the value $S(k)$ shown in (6) for each fractional frequency in the interval [0, 5].[12] Since there are 182 observations, this is equivalent to searching over θ in the interval 0 to 0.173. The frequency yielding the largest value of $S(k)$ is such that $k^* = 2.48$ and an associated value of

$S(k^*) = 61.68$. The *AIC* and *BIC* are −119.5 and −97.1, respectively. Since these values are lower than those from (15), as measured by the *AIC* and *BIC*, there is at least one frequency present in the regression residuals. We then used this frequency k^* to estimate a money demand function in the form:

$$m = \alpha_t + \alpha_1 p + \alpha_2 y + \alpha_3 r \quad (16)$$

where: $\alpha_t = a_0 + A_1^* \sin[\,2\pi(2.48)t/T\,] + B_1^* \cos[\,2\pi(2.48)t/T\,]$.

Table 5 reports these values along with the value of the *AIC* and *BIC* for the resulting regression. The table also reports the sample value of the *F*-statistic for the null hypothesis $A_1^* = B_1^* = 0$. The residuals from (16) were again standardized and the procedure was repeated. As shown in the second row of Table 5, the new value of $S(k^*)$ is 81.24 with a $k^* = 1.64$. We re-estimated the entire money demand equation including the two frequencies in α_t. We continued to repeat the process until we found no frequency that would reduce the *AIC* or the *BIC*. Since the sixth iteration increased the *BIC* (and, using Davis' critical values), produced a value of *sup* $S(k)$ that is not significant at conventional levels, we retained only the results from the first five iterations. The final estimate of the money demand relationship is:

$$\underset{}{m_t} = \alpha_t + \underset{(35.22)}{1.14p_t} + \underset{(19.23)}{0.891y_t} - \underset{(-7.11)}{0.005r_t} \quad (17)$$

where: $\alpha_t = a_0 + \sum_{i=1}^{5}\left[A_i^* \sin(2\pi k_i t/T) + B_i^* \cos(2\pi k_i t/T)\right]$

and: $a_0 = 0.685$ with a *t*-statistic of 1.63 and the A_i^* and B_i^* are given in Table 5. The *AIC* and *BIC* (incorporating the fact that two additional coefficients plus the frequency are estimated at each new iteration) steadily decline as the number of iterations increases through iteration 5.[13]

The final model fits the data quite well. As in (15), the price and income elasticities are of the correct magnitude. However, the interest rate semi-elasticity of demand for money now has the correct sign with a magnitude that is 7.1 times its standard error. The residuals are well-behaved. The last column of the table shows the *t*-statistic for the Engle-Granger (1987) cointegration test using the frequency components through iteration *i*. Notice that incorporating these frequency components enables us to reject a null hypothesis of no cointegration.[14] Figure 4 provides a visual representation of α_t. The striking impression is that the demand for money generally rose from 1959 through 1987. At this point, the demand for money suddenly declined. The decline continued through 1995 and then resumed its upward movement.

Another way to make the same point is to compare residuals (i.e., the 'equilibrium errors') from (15) and (17). As shown in Figure 5, the residuals from the Fourier model are only slightly better than those of the linear model over the first half of the sample period. The fact that the residuals of the linear model become highly persistent beginning in 1982 is consistent with the notion that (15) is not a cointegrating relationship. In contrast, the residuals of the Fourier model are not highly persistent and behave similarly throughout the entire sample period.

5.1 The Bootstrap

Supporting evidence for the significance of the selected trigonometric series can be gathered by testing the null hypothesis $\delta = 0$ in the following cointegrated system:

$$y_t = \mathbf{x}_t\beta + \mathbf{d}_t\delta + e_t \tag{18}$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mu_t \tag{19}$$

where: $\mathbf{x}_t$ is a vector of $I(1)$ exogenous variables, $\mathbf{d}_t$ is the vector containing the relevant sine and cosine terms in the Fourier expansion of the constant, e_t is the vector of residuals from the cointegrating regression and μ_t is a vector of $I(0)$ error terms. Small sample properties of inference on δ can at times be unsatisfactory (Li and Maddala, 1997) and bootstrapping methods have been proposed to improve such inference.

Generating bootstrap critical values for inference in cointegrated equations is, however, not straightforward. Bootstrapping the significance of the test statistic for δ = 0 in equation (18) using only the empirical distribution of error process e_t is inappropriate since it ignores the possibility that the errors may be autocorrelated and that the regressors in $\mathbf{x}_t$ might be endogenous in the sense that that the elements of $\boldsymbol{\mu}_t$ are correlated with e_t. Li and Maddala (1997) and Psaradakis (2001) introduced bootstrap procedures to be applied in this framework. Although they do not provide a formal proof, they present simulation evidence to establish that the bootstrap procedure introduced here achieves significantly improved small sample inference.[15]

A bootstrapping procedure allowing for autocorrelated residuals and endogeneity of $\mathbf{x}_t$ is performed according to the following steps (Psaradakis, 2001):

1. We estimate (18) and (19) using fully-modified least squares under the null hypothesis $\delta = \mathbf{0}$ to obtain a consistent estimate of β. The estimated model yields the residual estimates: $\hat{e}_t$ and $\hat{\boldsymbol{\mu}}_t$.

2. We draw bootstrap replications for the matrix of residuals $\hat{\boldsymbol{\mu}}_t^* = (\hat{e}_t, \hat{\boldsymbol{\mu}}_t)'$. To account for all possible autocorrelations and crosscorrelations, we estimate $\hat{\boldsymbol{\mu}}_t^*$ as the *VAR*(*p*) system:

$$\hat{\boldsymbol{\mu}}_t^* = \gamma_0 + \sum_{i=1}^{p} \gamma_i \hat{\boldsymbol{\mu}}_t^* + \boldsymbol{\varepsilon}_t \tag{20}$$

Resampling the estimated residuals from (20) yields the bootstrap estimates of $\hat{\boldsymbol{\mu}}_t^*$.

3. These bootstrap estimates are then used to construct the resampled values of $\mathbf{x}_t$ and y_t. Using the bootstrapped data, the model in (18) may be re-estimated and by repetition of this procedure the empirical distribution of the LR statistic for the null hypothesis δ = 0 may be built up and a *prob*-value derived.

When we preformed this procedure using the five frequency components reported in Table 5, we obtained a sample statistic with a *prob*-value of 0.000. As such, there is strong support for the claim that (17) forms a cointegrating relationship.

5.2 The error-correction model

In the presence of α_t, the four variables appear to form a cointegrating relationship; as such, there exists an error-correction representation such that m, y, p and r adjust to the discrepancy from the long-run equilibrium relationship. However, unlike a traditional error-correction model, adjustment will be nonlinear since the constant in the cointegrating vector is a function of time. As such, we estimated the following error-correcting model using the residuals from (17) as the error-correction term. Consider:

$$\Delta m_t = \underset{(-5.94)}{-0.207 ec_{t-1}} + \underset{(0.000)}{A_{11}(L)\Delta m_{t-1}} + \underset{(0.248)}{A_{12}(L)\Delta p_{t-1}} + \underset{(0.062)}{A_{13}(L)\Delta y_{t-1}} + \underset{(0.141)}{A_{14}(L)\Delta r_{t-1}} \tag{21}$$

$$\Delta p_t = \underset{(3.02)}{0.054 ec_{t-1}} + \underset{(0.742)}{A_{21}(L)\Delta m_{t-1}} + \underset{(0.000)}{A_{22}(L)\Delta p_{t-1}} + \underset{(0.306)}{A_{23}(L)\Delta y_{t-1}} + \underset{(0.254)}{A_{24}(L)\Delta r_{t-1}} \tag{22}$$

$$\Delta y_t = \underset{(1.75)}{0.091 ec_{t-1}} + \underset{(0.462)}{A_{31}(L)\Delta m_{t-1}} + \underset{(0.817)}{A_{32}(L)\Delta p_{t-1}} + \underset{(0.011)}{A_{33}(L)\Delta y_{t-1}} + \underset{(0.0030)}{A_{34}(L)\Delta r_{t-1}} \tag{23}$$

$$\Delta r_t = \underset{(1.45)}{0.676 ec_{t-1}} + \underset{(1.56)}{A_{41}(L)\Delta m_{t-1}} + \underset{(0.001)}{A_{42}(L)\Delta p_{t-1}} + \underset{(0.000)}{A_{43}(L)\Delta y_{t-1}} + \underset{(0.000)}{A_{44}(L)\Delta r_{t-1}} \tag{24}$$

where: ec_{t-1} = error-correction term (as measured by the residual from (17), $A_{ij}(L)$ = third-order polynomials in the lag operator L, parenthesis contain the t-statistic for the null hypothesis that the coefficient on the error-correction term is zero or the F-statistic for the null-hypothesis that all coefficients in $A_{ij}(L) = 0$, and constant terms in the intercepts are not reported.

Note that the money supply contracts and the price level increases in response to the previous period's deviation from the long-run equilibrium. However, income and the interest rate appear to be weakly exogenous.

5.3 The restricted model

One possible concern about the system given by (21) – (24) is money and the price level appear to be jointly determined endogenous variables. Moreover, income is weakly exogenous at the 5% significance level but not at the 10% level. With several jointly endogenous variables, the single-equation approach to examining a cointegrating relationship may be inappropriate unless a fully modified least squares procedure, such as that developed by Phillips and Hansen (1990), is used. For our purposes, it is convenient that the income and price elasticities of the money demand function are very close to unity. As such, it is possible for us to simply investigate the restricted money demand equation:

$$\underset{}{mp_t} = \underset{(-31.50)}{-0.425} + \underset{(2.53)}{0.005}r_t \qquad (25)$$

$AIC = -26.63 \quad BIC = -29.84$

where: mp_t = the logarithm of real money balanced divided by real GDP (i.e., $m_t - p_t - y_t$).

In (25), the interest rate is weakly exogenous and the money supply, price level and income level all appear in the left-hand-side variable $m_t - p_t - y_t$. This regression suffered the same problems as the unconstrained form of the money demand function. After applying our methodology to the constrained money demand function we obtained:

$$mp_t = \alpha(t) - \underset{(-3.82)}{0.003}r_t \qquad (26)$$

$AIC = -550.03 \quad BIC = -499.57$

and α_t = has the same form as (17).

The time path of α_t (not shown) is virtually identical to that shown in Figure 4. The error-correction model using the constrained form of the money-demand function is:

$$\Delta mp_t = \underset{(-6.98)}{-0.312 ec_{t-1}} + \underset{(0.000)}{A_{11}(L)\Delta mp_{t-1}} + \underset{(0.000)}{A_{12}(L)\Delta r_{t-1}} \quad (27)$$

$$\Delta r_t = \underset{(0.677)}{2.75 ec_{t-1}} + \underset{(0.098)}{A_{21}(L)\Delta mp_{t-1}} + \underset{(0.000)}{A_{22}(L)\Delta r_{t-1}} \quad (28)$$

where: ec_{t-1} = error-correction term (as measured by the residual from (23), $A_{ij}(L)$ = third-order polynomials in the lag operator L, parenthesis contain the t-statistic for the null hypothesis that the coefficient on the error-correction term is zero or the F-statistic for the null-hypothesis that all coefficients in $A_{ij}(L) = 0$, and intercepts are not reported.

5.4 Integer Frequencies

In order to illustrate the use of integer frequencies and to compare the approximation to that using continuous frequencies, we re-estimated the money demand function using discrete frequencies in the expanded interval [1, 8] so that θ ranges from 0.0345 to 0.241 in steps of 0.0345.

The results from estimating the money demand function with integer frequencies are shown in Table 6. The form is the same as that in (17) except that discrete frequencies 1, 2, 3, 4, 5 and 6 are used in the approximation for α_t. The bootstrap methodology need not be modified in any important way when using integer frequencies. As a group, these six integer frequencies are statistically significant at conventional levels. Although the fit (as measured by the *AIC* and *BIC*) is not as good as that using continuous frequencies, the Engle-Granger test strongly suggests that the residuals are stationary. The time-path of α_t using discrete frequencies (not shown) is nearly identical to that obtained using fractional frequencies.

5.5 Missing Variables

As suggested by Clements and Hendry (1998), a specification error resulting from an omitted variable can manifest itself in parameter instability. One major advantage of 'backing-out' the form of α_t is that it might help to suggest the missing variable responsible for parameter instability. Certainly, if a variable has the same time path as α_t, including it as a regressor would capture any instability in the intercept. In terms of our money demand analysis, the inclusion of a variable having the time profile exhibited in Figure 4 might suggest the form of the missing money. To demonstrate the point, we included a time trend in the demand for money function such that:

$$\alpha_t = a_0 + b_0 t + (a_1 + b_1 t)d_1 + (a_2 + b_2 t)d_2 \qquad (29)$$

where: $d_1 = 1$ for $1982{:}2 < t \leq 1995{:}2$ and 0 otherwise

$d_2 = 1$ for $t > 1995{:}2$ and 0 otherwise

Thus, instead of using our Fourier approximation, we represent α_t by a linear trend with breaks in the intercept and slope coefficients occurring at the time periods suggested by Figure 4. The estimated money demand function is:

$$m_t = \alpha_t + \underset{(18.62)}{0.807}p_t + \underset{(6.16)}{0.571}y_t - \underset{(-3.89)}{0.004}r_t \qquad (30)$$

$$\alpha_t = \underset{(3.79)}{2.49} + \underset{(6.57)}{0.008}t + (\underset{(21.14)}{1.65} - \underset{(-22.34)}{0.014}t)d_1 + (\underset{(-15.54)}{-1.04} + \underset{(9.45)}{0.004}t)d_2$$

$AIC = -476.71 \quad BIC = -447.88$

The Engle-Granger test indicates that the residuals from (30) are stationary: with four lags in the augmented form of the test, the t-statistic on the lagged level of the residuals is –5.07.

As measured by the *AIC* and *BIC,* this form of the money demand function does not fit the data quite as well as those using the Fourier approximation. Moreover, the price and income elasticities have been shifted downward. One reason for the superior fit of the Fourier model might simply be the fact that breaks in the time trend are actually smooth rather than sharp.

Although the Fourier approximations have better overall properties than (30), we used a trend-line containing two breaks for illustrative purposes only. The point is that a Fourier approximation can be used to 'back-out' the time-varying intercept. As such, the visual depiction of the time-varying intercept can be suggestive of a missing explanatory variable. Of course, in addition to a broken trend-line, there are other candidate variables. Figure 4 suggests that the large decline in wealth following Black Monday in October of 1987 might have been responsible for the decline in money demand. As stock prices recovered, the demand for M3 seemed to have resumed its upward trend. There does not seem to be enough data to determine whether the stock market decline following the events of 9 September 2001 had a similar effect on money demand.

6. Conclusion

In the paper, we developed a simple method that can be used to test for a time-varying intercept and to approximate its form. The method uses a Fourier approximation to capture any variation in the intercept term. As such, the issue becomes one of deciding which frequencies to include in the approximation. The test for a structural break works nearly as well as the Andrews and Ploberger (1994) optimal test if there is one break and can have substantially more power in the presence of multiple breaks. Perhaps, the most important point is that successive applications of the test can be used to 'back-out" the form of the time-varying intercept.

A number of diagnostic tests indicate that a linear autoregressive model of the U.S. inflation rate (as measured by the CPI) is inappropriate. It was shown that our methodology is capable of 'backing-out' the form of the nonlinearity. We also explored the nature of the approximation using an extended example concerning the demand for M3. Using quarterly U.S. data over the 1959:1 – 2004:2 period, we confirmed the standard result that the demand for money is not a stable linear function of real income, the price level and a short-term interest rate. The incorporation of the time-varying intercept resulting from the Fourier approximation appears to result in a stable money demand function. Moreover, the magnitudes of the coefficients are quite plausible and all are significant at conventional levels. The form of the intercept term suggests a fairly steady growth rate in the demand for M3 until late-1987. At that point, there was a sharp and sustained drop in demand. Money demand continued to decline until mid-1995 and then resumed its upward trend. The implied error-correction model appears to be reasonable in that money and the price level (but neither income nor the interest rate) adjust to eliminate any discrepancy in money demand.

There are a number of important limitations of the methodology. First, in a regression analysis, a structural break may affect the slope coefficients as well as the intercept. Our methodology forces the effects of the structural change to manifest itself only in the intercept term. A related point is that the alternative hypothesis in the test is that the residuals are not white-noise. It is quite possible that the methodology captures any number of departures from white-noise and places them in the intercept term. Third, we have not addressed the issue of out-of-sample forecasting. Although the Fourier approximation has very good in-sample properties, it is not clear how to extend the intercept term beyond the observed data. Our preference is to use an average of the last few values of α_t for out-of-sample forecasts. However, there are a number of other possibilities that are equally plausible. Anyone who has read the paper to this point can certainly add to the list of limitations. Nevertheless, we believe that the methodology explored in this paper can be useful for modeling in the presence of structural change.

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Table 1: Critical Values for the F^* Test with i.i.d. errors

	$T = 50$	$T = 100$	$T = 250$	$T = 1000$
	Maximum Frequency = $T/2$			
90%	5.81	6.37	7.17	8.53
95%	6.72	7.19	7.94	9.25
99%	8.87	9.09	9.72	10.95
	Maximum Frequency = $T/4$			
90%	4.95	5.61	6.44	7.80
95%	5.84	6.46	7.19	8.53
99%	8.08	8.31	8.92	10.21
	Maximum Frequency = $T/12$			
90%	3.54	4.27	5.19	6.69
95%	4.38	5.09	5.94	7.44
99%	6.27	6.95	7.65	9.17

Table 2: Power of the Andrews-Ploberger and Davies Tests with One Break

Andrews	δ **= 0**	δ **= 0.5**	δ **= 1**
1%	0.008	0.115	0.652
5%	0.043	0.274	0.825
10%	0.094	0.399	0.896
Davies	δ **= 0**	δ **= 0.5**	δ **= 1**
1%	0.007	0.105	0.585
5%	0.047	0.290	0.794
10%	0.096	0.409	0.891
Table 1: Reports size (δ = 0) and power statistics for Andrews and Davies test applied to the process in (11 and 12). Significance evaluated by means of bootstrap.			

Table 3: Power of the Andrews-Ploberger and Davies Tests with Two Breaks

Andrews	δ **= 0**	δ **= 0.5**	δ **= 1**
1%	0.008	0.026	0.103
5%	0.043	0.103	0.294
10%	0.094	0.185	0.443
Davies	δ **= 0**	δ **= 0.5**	δ **= 1**
1%	0.007	0.074	0.444
5%	0.047	0.213	0.671
10%	0.096	0.335	0.772
Table 2: Reports size (δ = 0) and power statistics for Andrews and Davies test applied to the process in (14). Significance evaluated by means of bootstrap.			

Table 4: Results of the Dickey-Fuller Tests

Variable	Lags	t-statistic
Δm	1	-1.20
Δy	2	-3.03
Δp	3	-1.62
Δr	3	-1.67

Critical values for the Dickey-Fuller test are -3.99, -3.43 and -3.13 at the 1%, 5% and 10% significance levels, respectively. Since the test for the interest rate did not include a deterministic time trend, the appropriate critical values for the Δr equation are -3.46, -2.88 and -2.57 at the 1%, 5% and 10% significance levels, respectively.

Table 5: Results of the Successive Iterations

Iteration	$S(k^*)$	k_i^*	*AIC*	*BIC*	A_i^*	B_i^*	$F(k^*)$	*t*
1	61.69	2.48	–119.5	–97.1	–0.003 (0.0056)	–0.058 (0.0057)	51.88	–2.07
2	81.24	1.64	–336.9	–304.8	–0.080 (0.0039)	0.043 (0.0088)	208.77	–3.54
3	72.32	3.73	–511.4	–469.8	0.009 (0.0019)	–0.039 (0.0024)	145.16	–4.55
4	45.79	4.75	–572.3	–521.1	–0.004 (0.0016)	–0.014 (0.0017)	38.14	–5.56
5	2.48	5.00	–596.8	–535.9	0.003 (0.0038)	–0.028 (0.0051)	15.47	–5.75
6	0.03	4.24	–592.6	–522.1	0.009 (0.0155)	0.012 (0.0158)	0.87	–5.84

NOTE: Davies critical values for $S(k^*)$ are 10.58, 12.09, 13.59 and 15.55 at the 10%, 5%, 2.5% and 1% significance levels, respectively. We do not bootstrap the $S(k^*)$ statistic for the individual frequency components. Standard errors of the estimated coefficients are in parentheses. Critical values for the $F(k^*)$ test are 10.63, 7.78 and 6.59 at the 1%, 5%, and 10% significance levels, respectively. t is the sample value of the Engle-Granger statistic for the null hypothesis of non-stationarity of the regression residuals.

Table 6: The Approximation with Discrete Frequencies

Iteration	$S(k^*)$	k_i^*	*AIC*	*BIC*	A_i^*	B_i^*	$F(k^*)$	*t*
1	55.33	3	–108.3	–85.9	–0.026 (0.0065)	0.048 (0.0060)	43.27	–2.81
2	48.90	2	–210.7	–178.6	0.049 (0.0054)	0.044 (0.0046)	70.80	–4.52
3	44.84	1	–435.9	–394.2	0.084 (0.0119)	–0.049 (0.0058)	220.40	–3.82
4	50.21	5	–506.2	–454.9	–0.014 (0.0021)	–0.013 (0.0020)	44.24	–4.90
5	33.08	6	–547.8	–486.9	0.005 (0.0012)	0.012 (0.0012)	25.14	–5.56
6	26.76	4	–625.9	–555.4	–0.015 (0.0020)	–0.014 (0.0017)	48.75	–6.41
7	1.78	7	–623.0	–542.9	–0.002 (0.0017)	–0.002 (0.0015)	1.39	–6.65

NOTE: See notes for Table 5

Endnotes

[1] Let the function α_t have the Fourier expansion:

$$\alpha_t = \alpha_0 + \sum_{k=1}^{\infty}\left[A_k \sin\frac{2\pi k}{T}\bullet t + B_k \cos\frac{2\pi k}{T}\bullet t \right]$$

and define $F_s(t)$ to be the sum of the Fourier coefficients:

$$F_s(t) = \sum_{k=1}^{s}\left[A_k \sin\frac{2\pi k}{T}\bullet t + B_k \cos\frac{2\pi k}{T}\bullet t \right]$$

Then, for any arbitrary positive number h, there exists a number N such that:

$| \alpha_t - F_s(t) | \leq h$ for all $s \geq N$.

[2] Since the approximation works extremely well, even for a sample size of 16, we use only the approximate forms of the test statistic. Also note that θ need not be chosen such that k is an integer; in fact, below we illustrate that fractional values of k can provide good approximations to changes in the conditional mean of a series.

[3] The Andrews-Ploberger test is only included for illustrative purposes--it is well known that it is not the optimal test for a double break.

[4] If T is large, the assumption of the known variance is overly strong; the asymptotic results go through using the estimated variance.

[5] The *AIC* select the 12-lag specification while the *BIC* selects a model with 11-lagged changes. The essential results are virtually identical using either specification.

[6] The results are similar if we use fractional frequencies.

[7] Since we searched over the various frequencies to find the best fit, a number of the t-statistics we report do not have their usual interpretation.

[8] The point of this section was to illustrate Davies test for a structural break. When we applied the Davies test to the standardized residuals of (14), we obtained $k^* = 2.0$ and $S(k^*) = 4.95$. Using Davies' critical values and the bootstrapped critical values, the second frequency was not significant at conventional levels. By way of comparison, the successive applications of the AP test indicated only a one-time shift in the intercept occurring in 1967:5

[9] Note that this method selects the identical frequencies as using the OLS-based *Trig*-test or the method suggested by Enders and Lee (2004) for their $F(k^*)$ statistic.

[10] Note that the Enders and Lee (2004) critical values are not directly applicable to our study of the money demand function. Their critical values are derived from a univariate framework and not from a cointegrated system. Nevertheless, the fact that there is a distribution of the unit-root case suggests that there is a distribution for the case of cointegrated variables.

[11] Almost identical results to those reported below hold if we use M2 instead of M3.

[12] We used a maximum value of $k = 5$ since we wanted to consider only 'low frequency' changes in the intercept. Also note that we searched at intervals of 1/512. The results turn out to be similar if we use integer frequencies.

[13] Also shown in Table 5 is the sample value of $F(k^*)$. It is interesting to note that these values of $F(k^*)$ exceed the critical values reported by Enders and Lee (2004) through iteration 5.

[14] It is not our intention here to provide a new test for cointegration. Note that the critical values for the Engle-Granger test may depend on the inclusion of the frequency components. After all, the frequency components were chosen by means of a grid search so as to provide the component with the best fit. A proper cointegration test would bootstrap the critical of the Engle-Granger test statistic. However, that would take us far beyond the purpose of this paper.

[15] To the best of our knowledge, no theoretical arguments are available yet, to establish whether this, or any other bootstrap procedure, generates consistent inference in the context of cointegrated regressions.

Figure 1: Four Fourier Approximations to Changes in the Mean

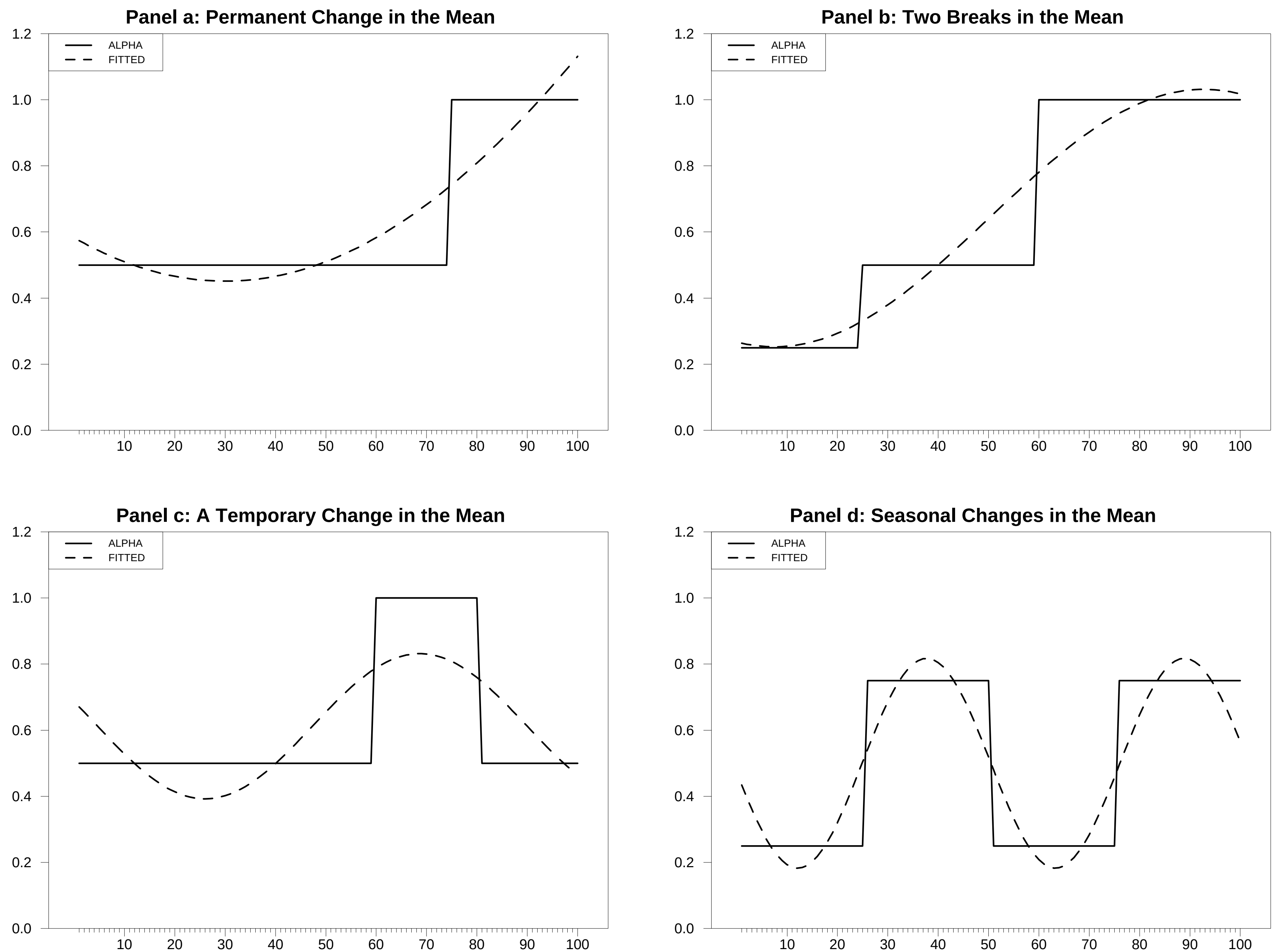


Figure 2: Increasing the Number of Freqencies

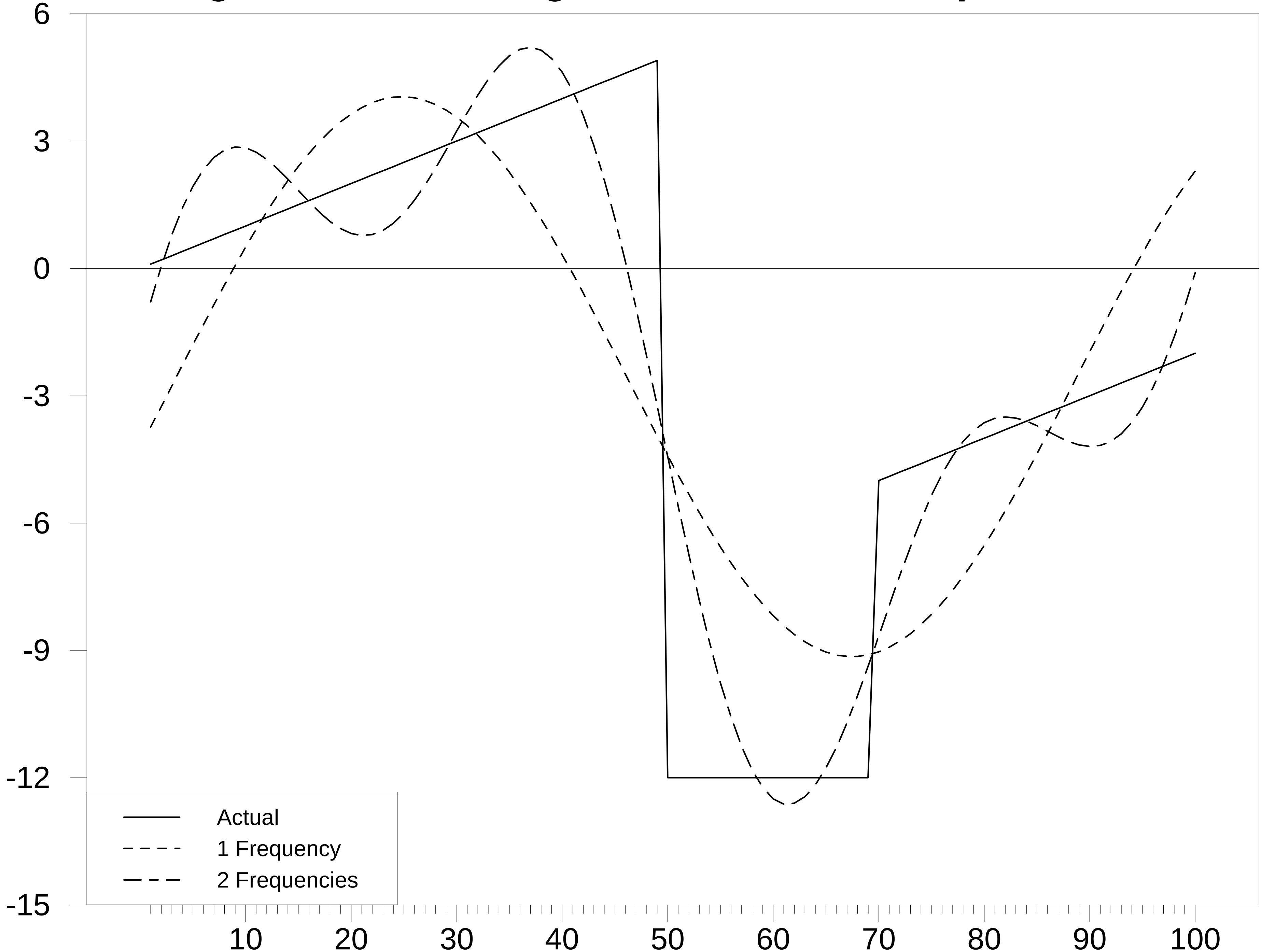
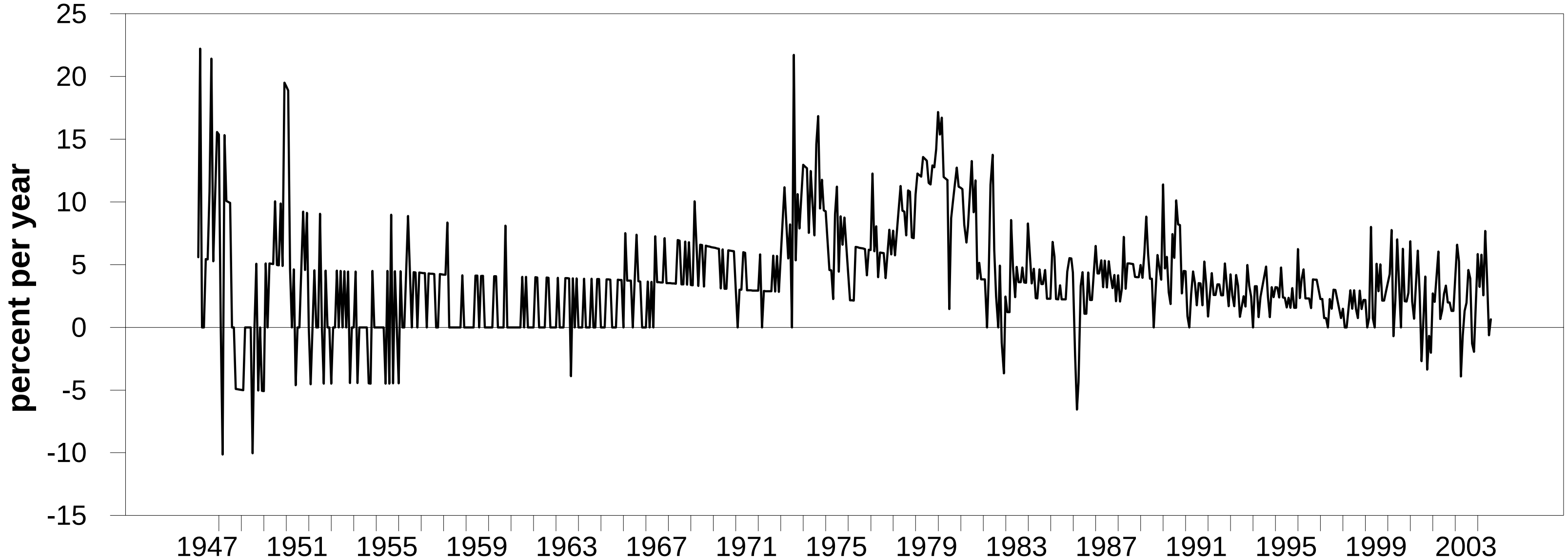


Figure 3: A Structual Break in U.S. Inflation?

Panel a: The U.S. Inflation Rate



Panel b: The Time-Varying Coefficient

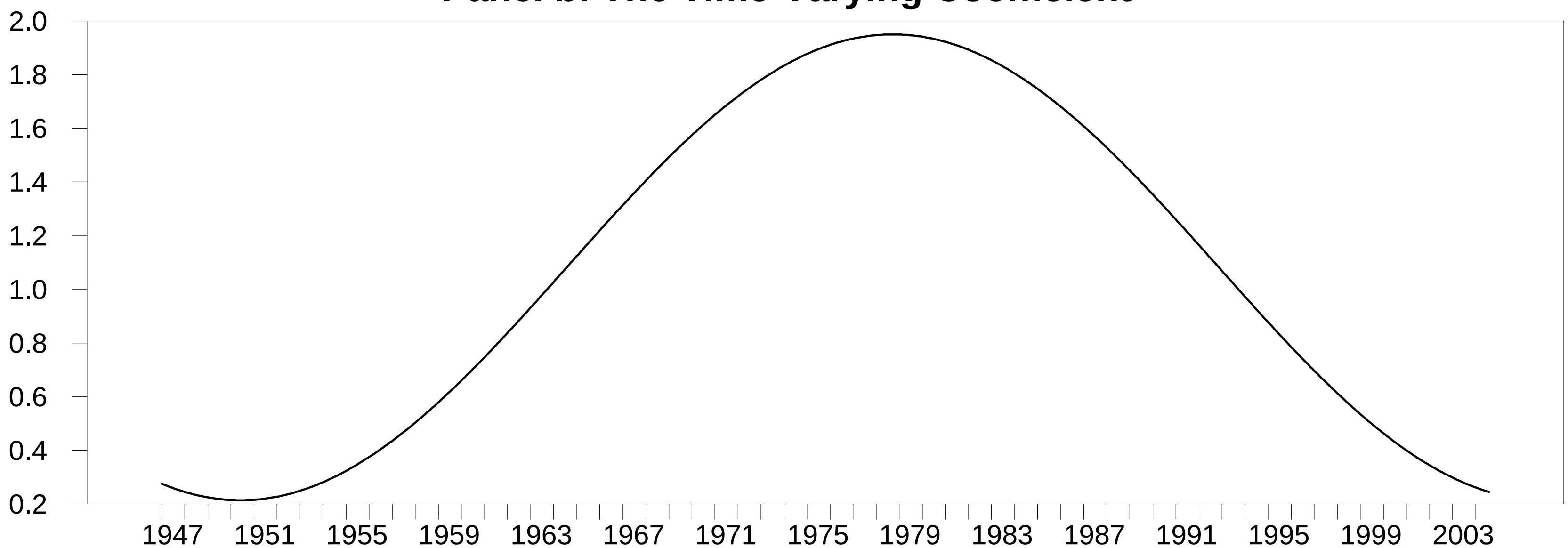


Figure 4: Intercept of the Demand for Money

(5 Frequencies)

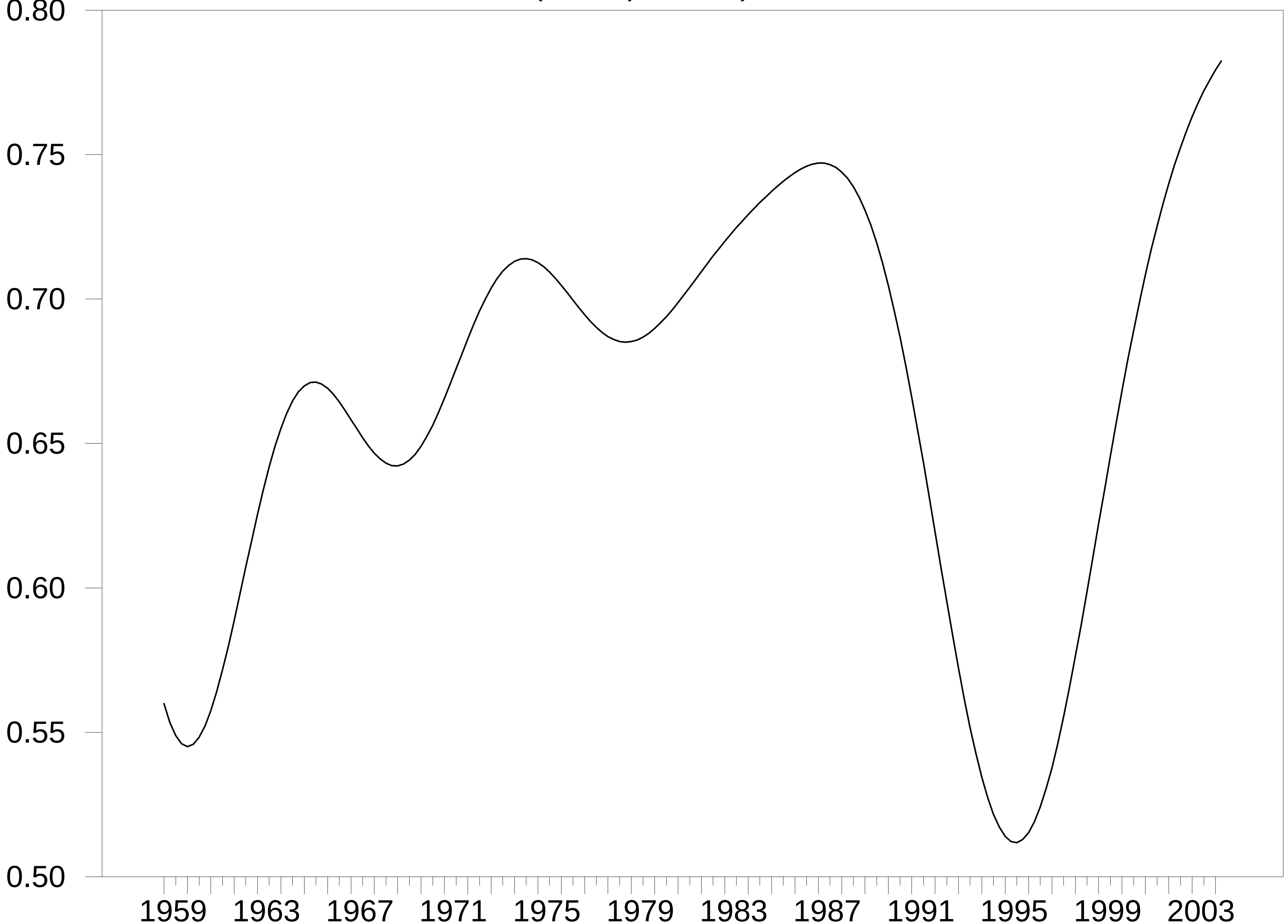


Figure 5: Equilibrium Errors from the Linear and Fourier Models